

Fixed Point Iterative Methods à la Fujimoto

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Abstract

Fujimoto's classical approximation technique (independently developed in Brunton (1973) and Fujimoto and Nishiwaki (1982)) is an iterative method to fold one side of the paper into n equal parts. Given an initial guess for one of the target division marks, the process produces new creases on the paper by bringing the left or the right end of the side of the paper onto the most recent crease. As the folding goes on, the creases define sequences, each one converging monotonously to its corresponding division mark. The method works for any initial guess, because each folding step halves the error of the approximation. The limit marks are, indeed, fixed points of the return map.

Fujimoto's method presents three outstanding practical features for origami purposes (mainly, finding landmarks), distinguishing it from other (exact or approximate) methods of division:

- (i) The folding manoeuvre is simple: carrying a point onto another is far more accurate and easy than folding the line connecting two given points (particularly if they are distant).
- (ii) It is a clean method, because it just performs some small marks in the border of the paper, leaving the whole interior of the paper untouched.
- (iii) The rule to know which end of the paper should be carried to the last crease is very simple: **imagine that the last approximation is exact** and just choose the side with an hypothetically even number of divisions, so that by **performing the move, the crease obtained would be one of the desired target division points**. In other words, the final (limit) division is fixed by the folding moves. We shall call those moves *fixing moves*.

We are interested in iterative folding methods presenting those three features. In fact, we have successfully applied fixing moves to division problems beyond the one solved by Fujimoto's method. The simplest example is shown in Fig. 1. By repeatedly applying steps 1 and 2, one gets two sequences of approximations $x_n \xrightarrow{n \rightarrow \infty} X$ and $y_n \xrightarrow{n \rightarrow \infty} Y$. Both limit points divide their sides into two segments in proportion $1 : \sqrt{2}$. Notice that both X and Y are fixed by the moves of steps 1 and 2; e.g. taking B to X yields Y . The best known Fujimoto's method for division in thirds shares the same left-right folding scheme. Indeed, we have solved a similar problem: we have found X and Y so that the segments \overline{AX} , \overline{XY} and \overline{YB} are equal. Another way to look at this phenomenon is regarding both divisions as the solution of the circle-packing problems shown

in Fig. 2. Notice that the proportion $1 : \sqrt{2}$ can be determined by a kite fold, but the method shown achieves it without leaving (perhaps) unwanted creases in the interior of the paper.

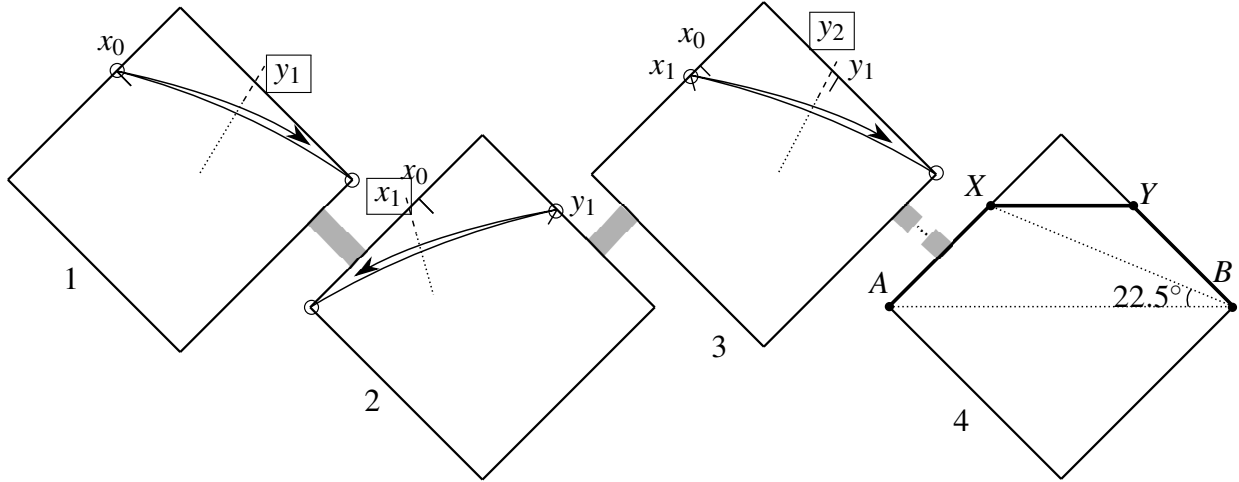


Figure 1: Iterative method to divide the side of the square in proportion $1 : \sqrt{2}$.

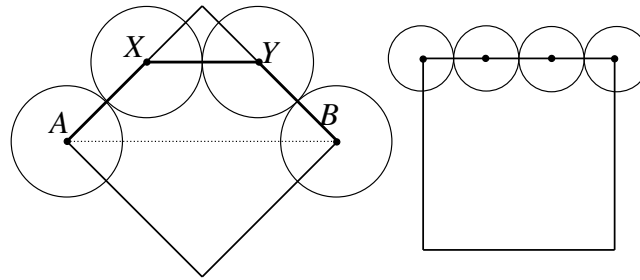


Figure 2: Both methods find the solution of a circle-packing problem.

In this work we construct new iterative sequences with the nice features of Fujimoto's method combining fixing moves. We describe a wide family of examples, including iterative methods to divide the side of the square in proportion $n : m\sqrt{2}$ for every $n, m \in \mathbb{N}$, or to inscribe several equilateral polygons in a square, including the pentagon of Montroll's five-sided square. We also apply the usual fixed-point methods machinery (mainly, Banach's Fixed Point Theorem) to study the convergence of our methods from the numerical point of view. The behaviour of these methods differs from that of Fujimoto's: some sequences will be oscillative, and the error decay is not uniform (it's typically faster in the beginning). In some cases, the convergence is quicker than with Fujimoto. Some methods will require two initial guesses to be launched.

References

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